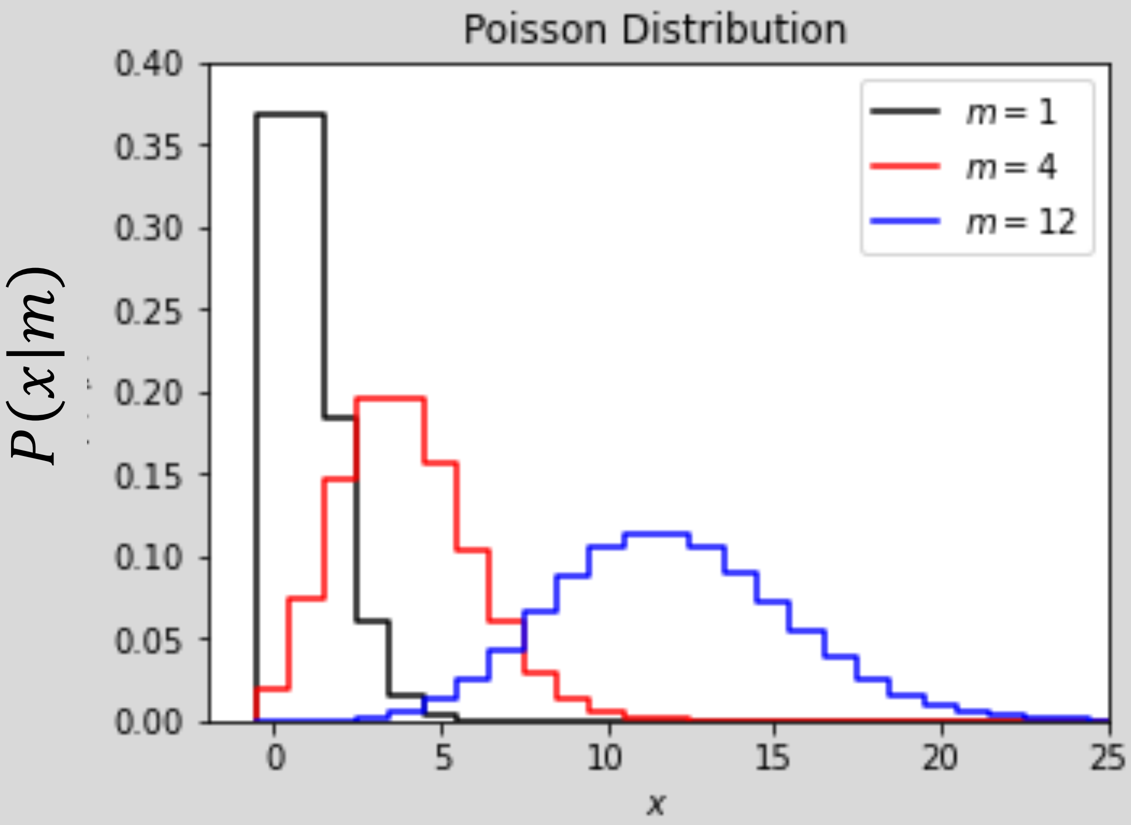
Photon Statistics and the Poisson distribution

For a series of discrete random events (like photons hitting a detector), the probability of seeing ***x*** events given an expectation of ***m* events** is given by the **Poisson distribution *P_x***

$$P(x|m) = \frac{m^x e^{-x}}{x!}$$

In astronomy terms:

Think of a star whose brightness is such that you'd expect to get **m** photons per second hitting your detector. What is the probability that you will measure **x** photons in a one second exposure?



Photon Statistics

As **m** (the **expectation value**) gets large, the distribution resembles a **Gaussian** or **normal** distribution.

$$dP = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

The **variance** of any distribution is defined as

$$\sigma^2 \equiv \frac{1}{n}\sum (x_i - m)^2$$

In the general Gaussian distribution, σ is independent from m.

But for the Poisson distribution, **σ^2=m**.

Terminology:
σ^2 = "variance" (np.var)
σ = "standard deviation" (np.std)

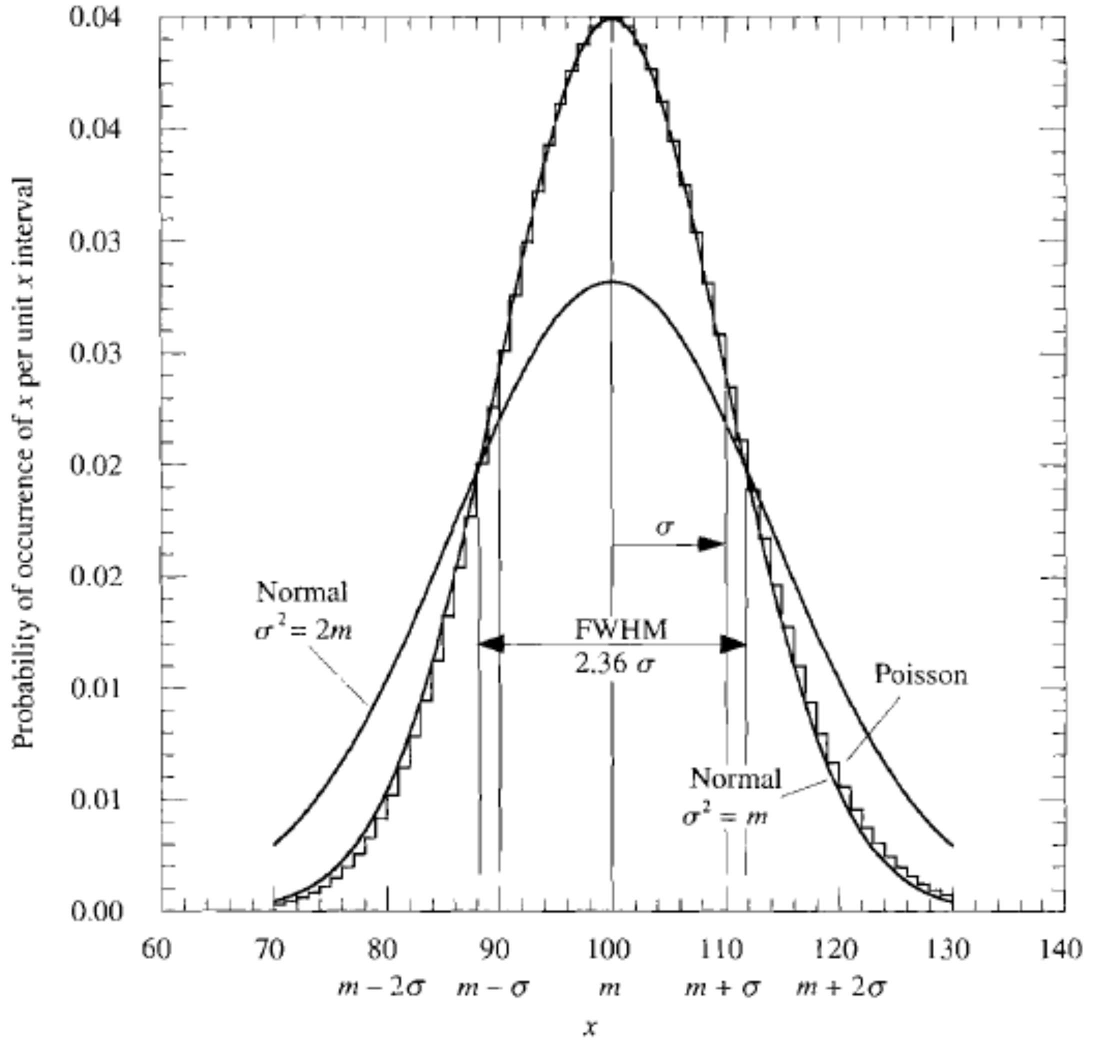
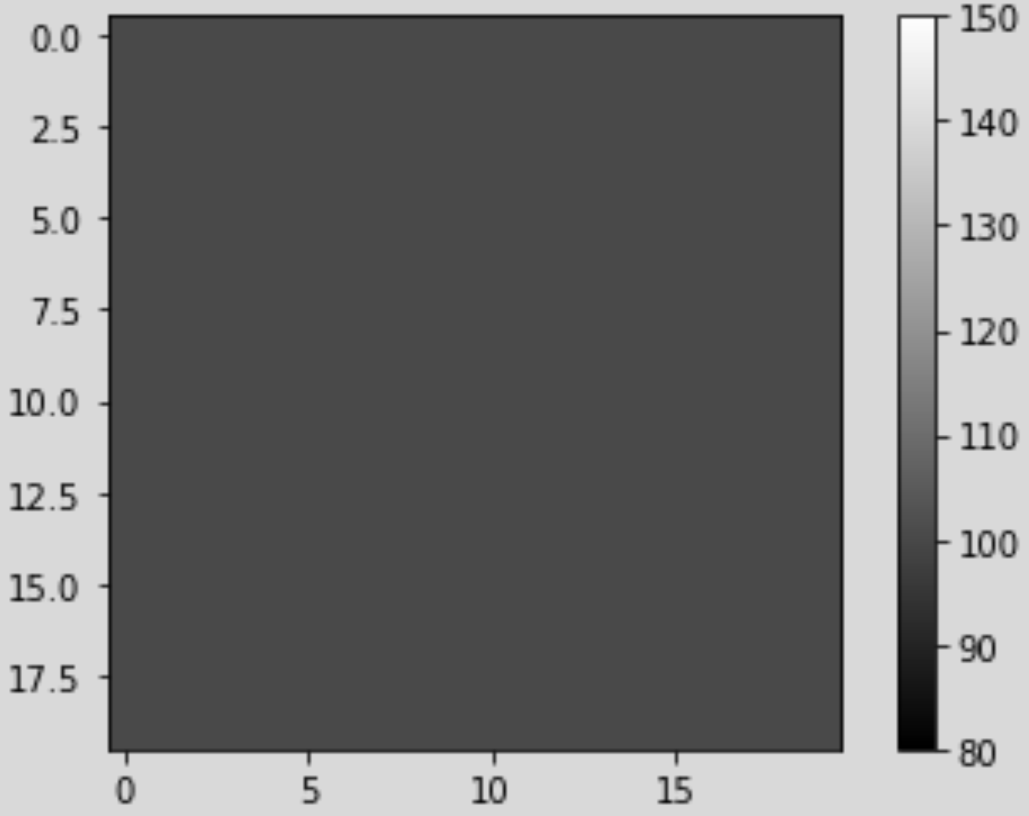


Figure 6.8. The Poisson (step curve) and normal distributions (smooth curves) for the mean value $m = 100$. The normal distribution is given for two values of the width parameter σ_w which is shown in the text to be equal to the standard deviation σ. The Poisson distribution approximates well the normal distribution if the latter has $\sigma = \sqrt{m}$. Note the slight asymmetry of the Poisson distribution relative to the normal distribution. The standard deviation and full width half maximum widths are shown for the higher normal peak; the two normal curves happen to cross at the FWHM point.

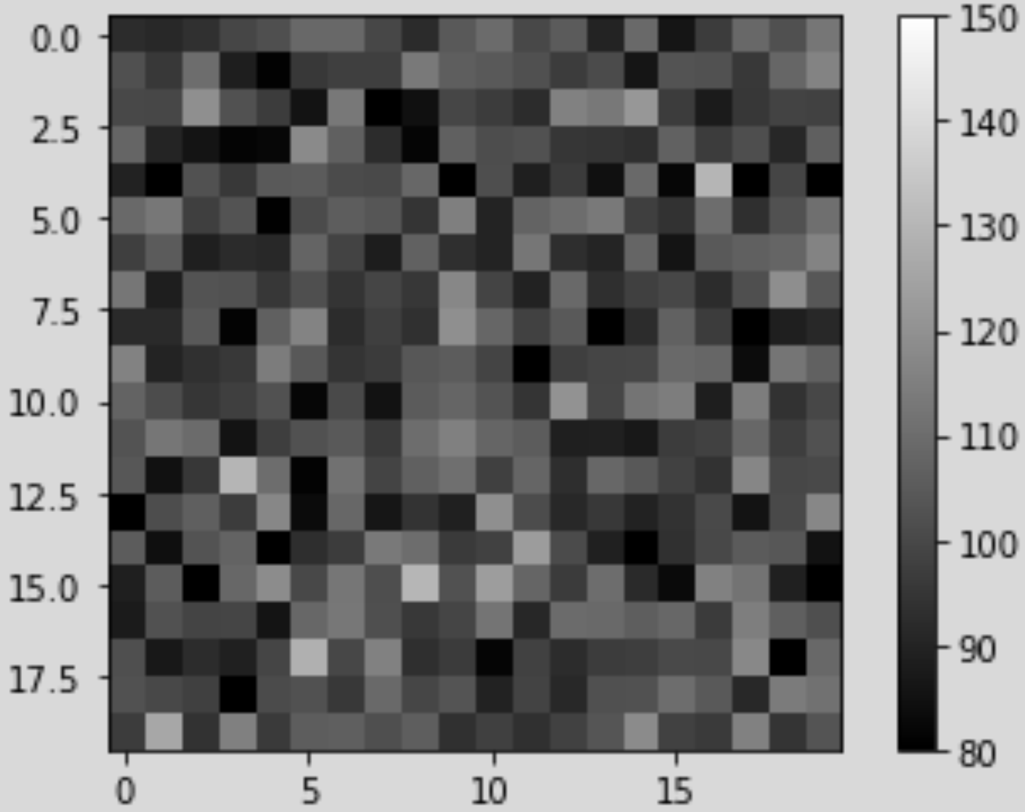
Detection significance

Say the background sky gives m=100 ***photons*** per pixel. By Poisson stats, the uncertainty in the sky level is then $\sigma = \sqrt{m} = \sqrt{100} = 10$ photons. So the sky level is 100 ± 10 photons/pixel.



Detection significance

Say the background sky gives m=100 ***photons*** per pixel. By Poisson stats, the uncertainty in the sky level is then $\sigma = \sqrt{m} = \sqrt{100} = 10$ photons. So the sky level is 100 ± 10 photons/pixel.

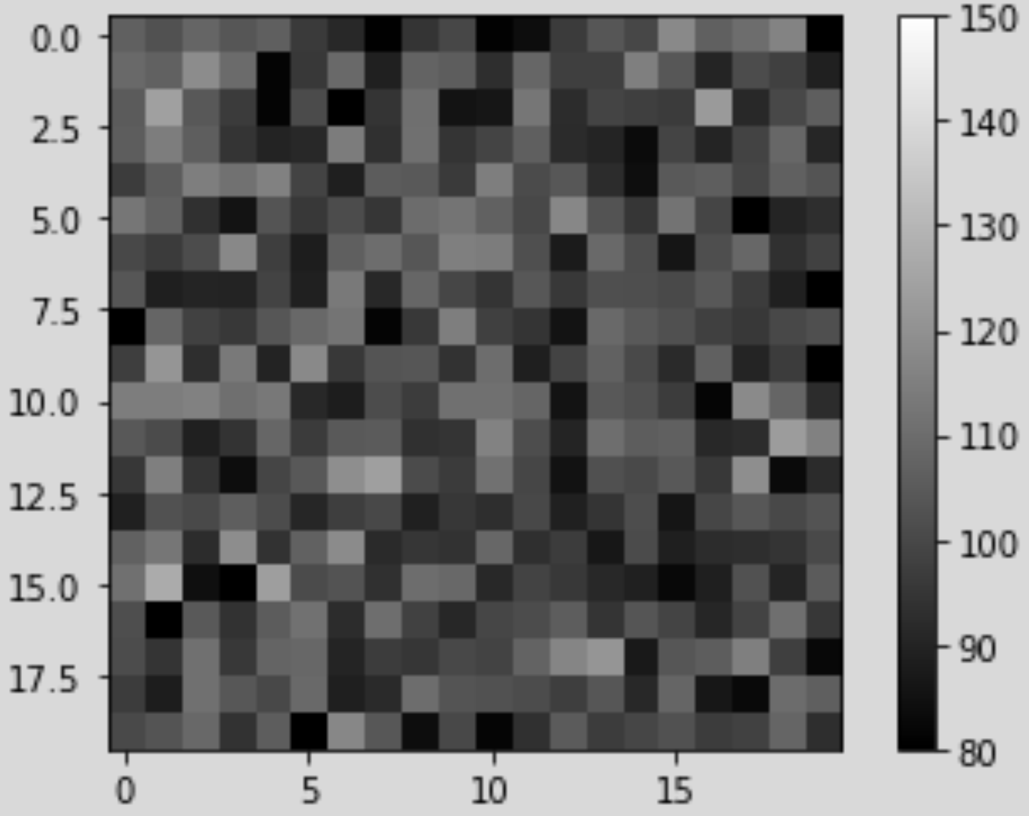


Detection significance

Say the background sky gives m=100 ***photons*** per pixel. By Poisson stats, the uncertainty in the sky level is then $\sigma = \sqrt{m} = \sqrt{100} = 10$ photons. So the sky level is 100 ± 10 photons/pixel.

How faint of a (one pixel) star could you detect?

N_* = 10 photons ➔ 1σ detection, quite possibly just a sky fluctuation. No detection.



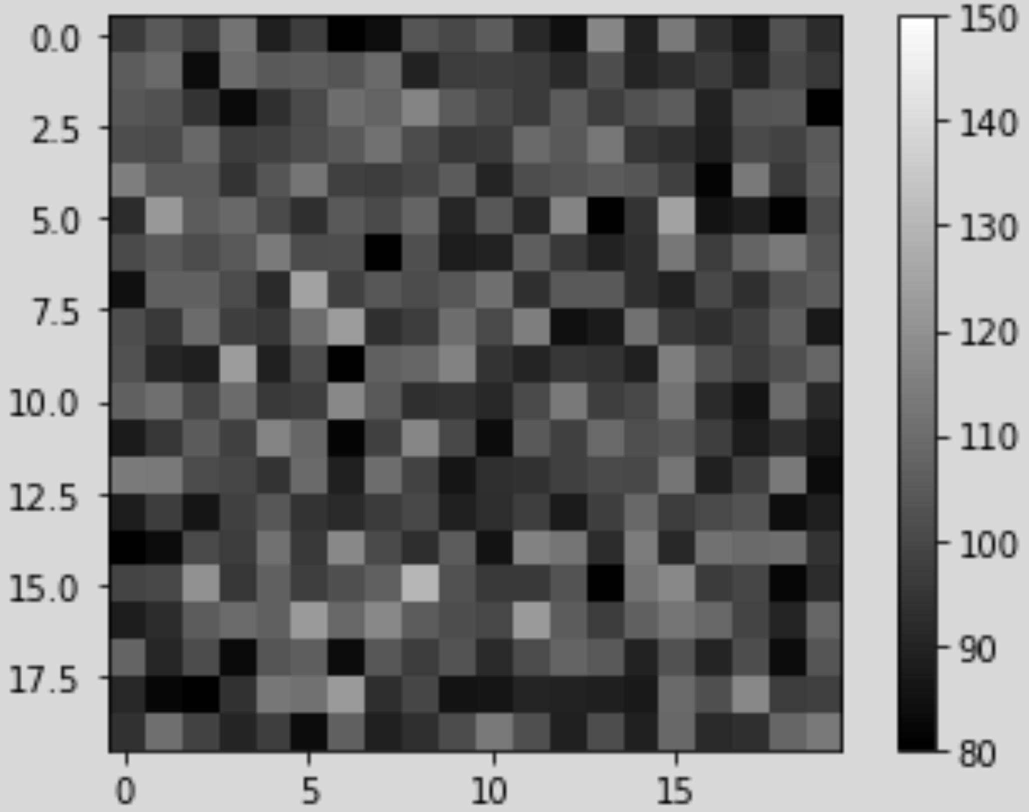
Detection significance

Say the background sky gives m=100 ***photons*** per pixel. By Poisson stats, the uncertainty in the sky level is then $\sigma = \sqrt{m} = \sqrt{100} = 10$ photons. So the sky level is 100 ± 10 photons/pixel.

How faint of a (one pixel) star could you detect?

N_* = 10 photons ➔ 1σ detection,
quite possibly just a sky fluctuation.
No detection.

N_* = 30 photons ➔ 3σ detection,
likelihood of such a sky fluctuation is small.
Borderline detection.



Detection significance

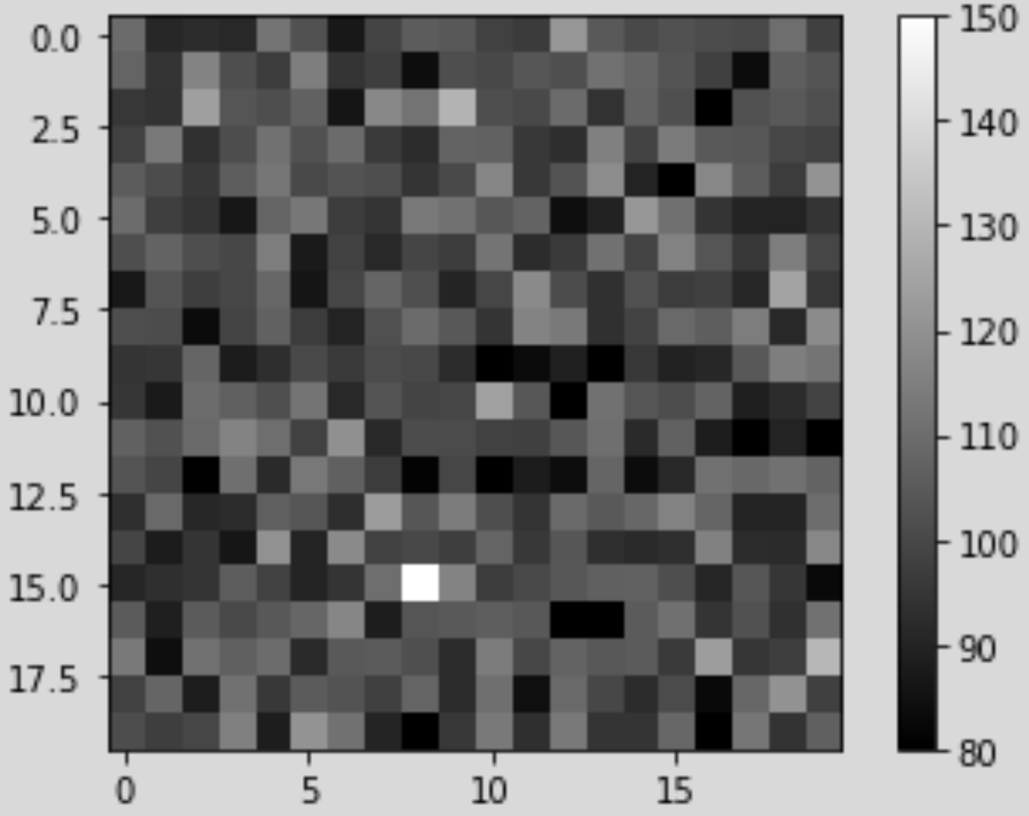
Say the background sky gives m=100 ***photons*** per pixel. By Poisson stats, the uncertainty in the sky level is then $\sigma = \sqrt{m} = \sqrt{100} = 10$ photons. So the sky level is 100 ± 10 photons/pixel.

How faint of a (one pixel) star could you detect?

N_* = 10 photons ➔ 1σ detection,
quite possibly just a sky fluctuation.
No detection.

N_* = 30 photons ➔ 3σ detection,
likelihood of such a sky fluctuation is small.
Borderline detection.

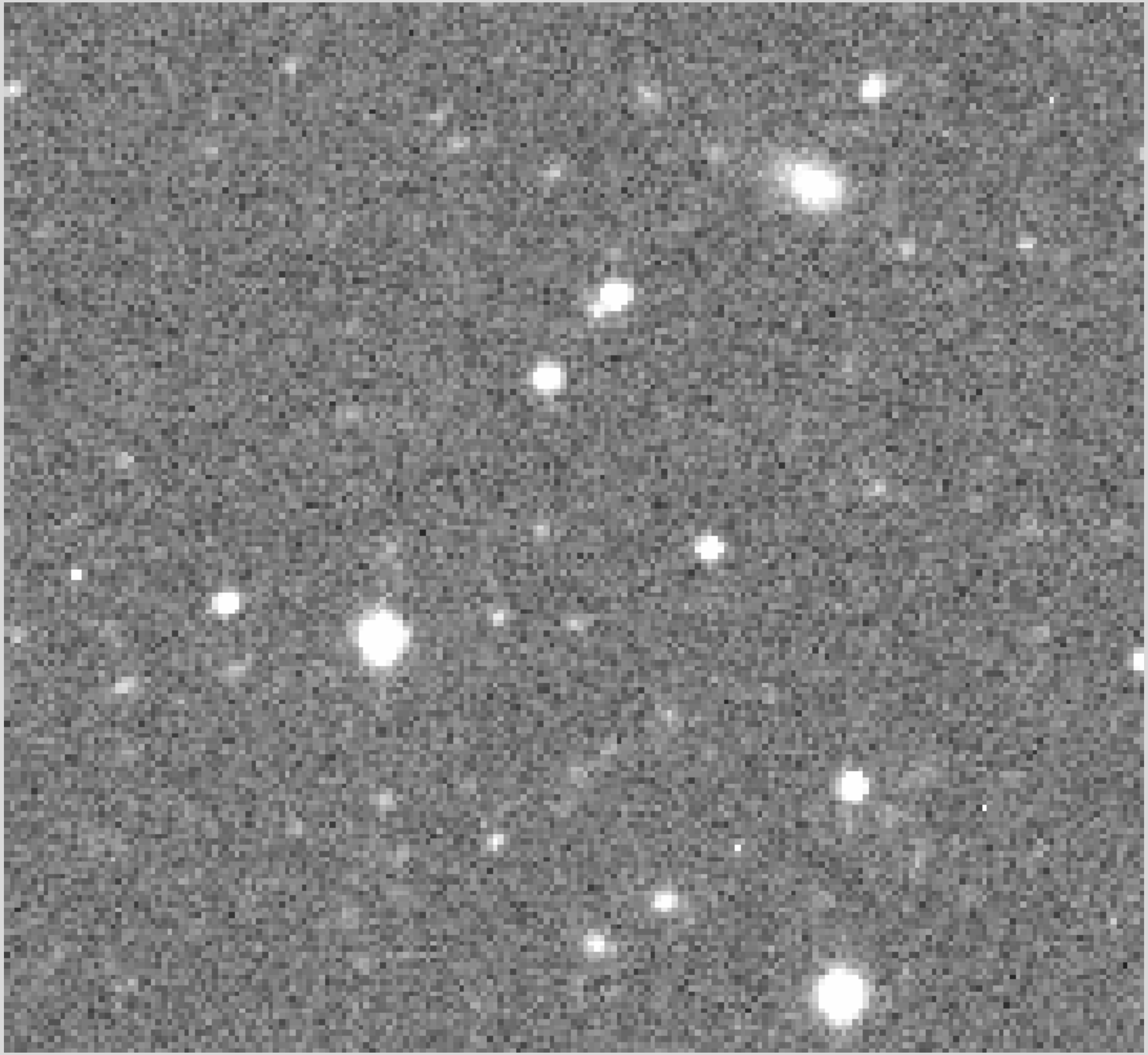
N_* = 100 photons ➔ 10σ detection,
likelihood of a sky fluctuation is vanishingly tiny.
Strong detection.



But, reality:

1. Stars (and galaxies!) are spread over many pixels, not just one. We have to think about integrating up the flux in some aperture, and then correcting for the sky flux in that aperture.

2. Photon noise from the sky isn't the only noise source. Need to also worry about:
   - Photon noise from the star
   - Readout noise from the detector
   - (Maybe) Dark noise: thermal electrons in the detector



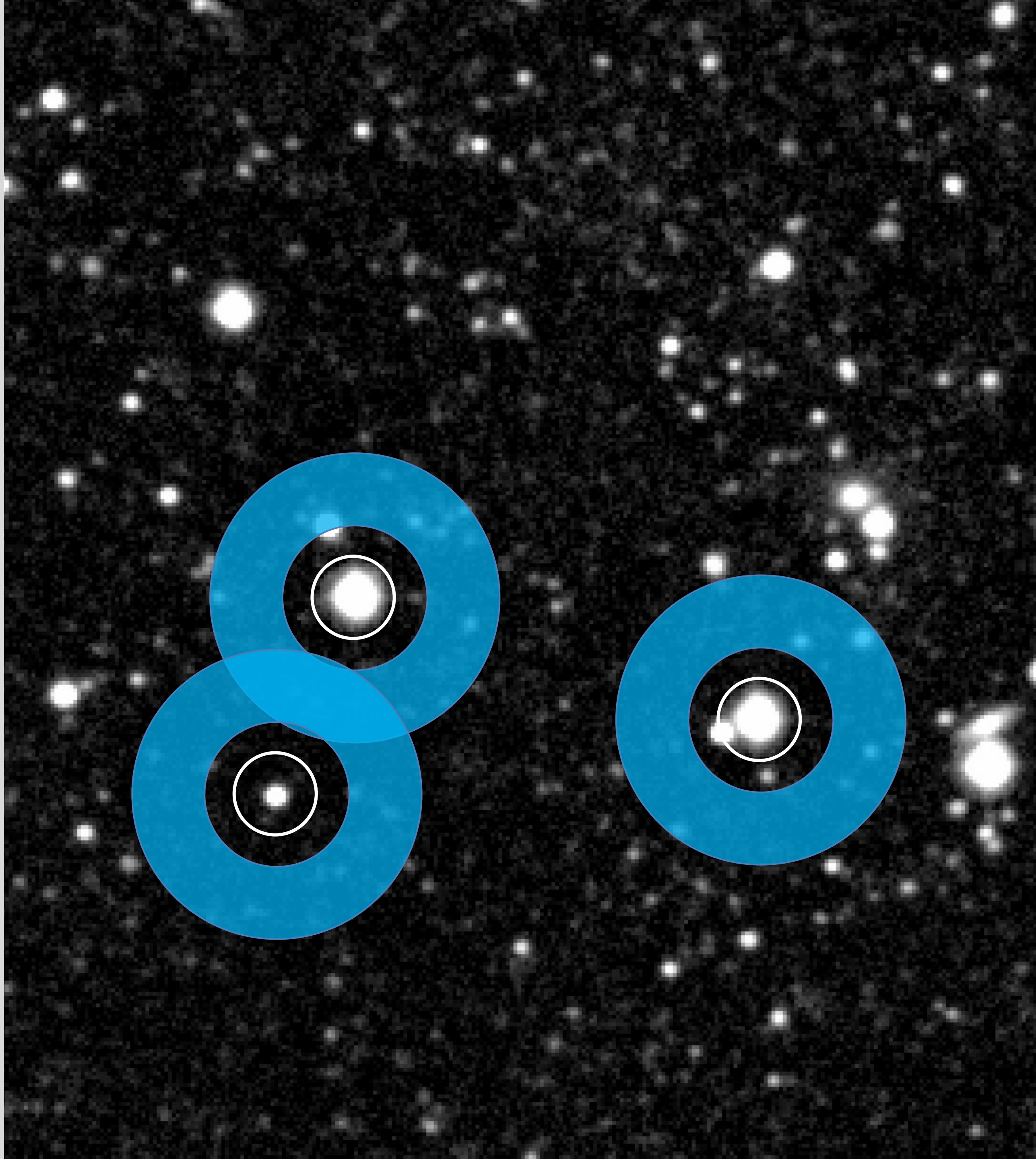
Aperture Photometry (Stars)

Measure flux (total counts) inside aperture of given size r, which contains n_{pix} pixels.

Estimate sky flux level (ADU/pix) from "average" ADU in pixels in surrounding annulus.

$$f = \left(\sum I_{aper}\right) - \langle I_{sky} \rangle \times n_{pix}$$

$$m_{inst} = -2.5 \log f + C$$



Measuring Signal-to-Noise: Detection quality

Consider measuring the flux from a star inside an aperture that contains n_{pix} pixels.

Signal:

- N_*, the total number of photons from the star.

Noise:

- Total Poisson noise from the star: $\sigma = \sqrt{N_*}$
- Per-pixel Poisson noise from the sky: $\sigma = \sqrt{N_S}$
- Per-pixel Poisson noise from dark current: $\sigma = \sqrt{N_D}$
- Per-pixel CCD read noise: $\sigma = N_R$

These N's all refer to photons or electrons (e^-), not detector counts (ADU)!

If needed, use gain (e^-/ADU) to convert counts to e^-

*Read noise is **not** photon statistics, so it doesn't get square-rooted. The read noise level is what you measure from the zero images.*

These noise contributions add in quadrature, so we get:

$$\frac{S}{N} = \frac{N_*}{\sqrt{N_* + n_{pix}(N_S + N_D + N_R^2)}}$$

"The CCD Equation"

see Howell, Chapter 4.4

Example: Schmidt Telescope + CCD

- gain = 2.5 e⁻/ADU
- read noise = 3.6 e⁻
- N_D = 0 ADU

$$\frac{S}{N} = \frac{N_*}{\sqrt{N_* + n_{pix}(N_s + N_D + N_R^2)}}$$

In a 60s exposure in the M filter, we get

- Sky = 80 ADU = 200 photons ($\pm\sqrt{200} = 14$) per pixel
- So let's say that inside a circular aperture of r=5 pixels, a star has 136,000 ADU, or 340,000 photons. Since the aperture contains $n_{pix} \approx \pi 5^2 \approx 80$ pixels, we calculate:

$$\frac{S}{N} = \frac{340{,}000}{\sqrt{340{,}000 + 80\times(200 + 0 + 3.6^2)}} = 570$$

For a fainter star that produces ≈ 700 ADU, the same calculation gives S/N ≈ 12.

Uncertainty in Magnitude (*for small uncertainties,* σ):

- $\sigma_m \approx \left(\sigma_f/f\right) \approx (S/N)^{-1}$
- Star 1: S/N = 570, so $\sigma_m \approx$ 0.002 mag
- Star 2: S/N = 12, so $\sigma_m \approx$ 0.09 mag

Very important*: This calculation refers to random error in the measurement; calibration uncertainties set a floor to the final photometric uncertainty. It is very hard to do photometry with a true accuracy better than 0.01−0.02 mag (1% − 2% uncertainty)*

S/N scaling with exposure time: how does S/N change as I expose longer?

$$\frac{S}{N} = \frac{N_*}{\sqrt{N_* + n_{pix}(N_s + N_D + N_R^2)}}$$

Case 1: “Detector limited” (Faint things)

Detector noise (N_R) dominates the counts, so $S/N \approx N_*/N_R$
Since N_R is independent of exposure time, $S/N \propto N_* \propto t_{exp}$

Case 2: “Source limited” (Bright things)

Photons from the star (N_*) dominate the counts, so $S/N \approx N_*/\sqrt{N_*} \approx \sqrt{N_*}$
Since N_* scales with exposure time, $S/N \propto \sqrt{t_{exp}}$

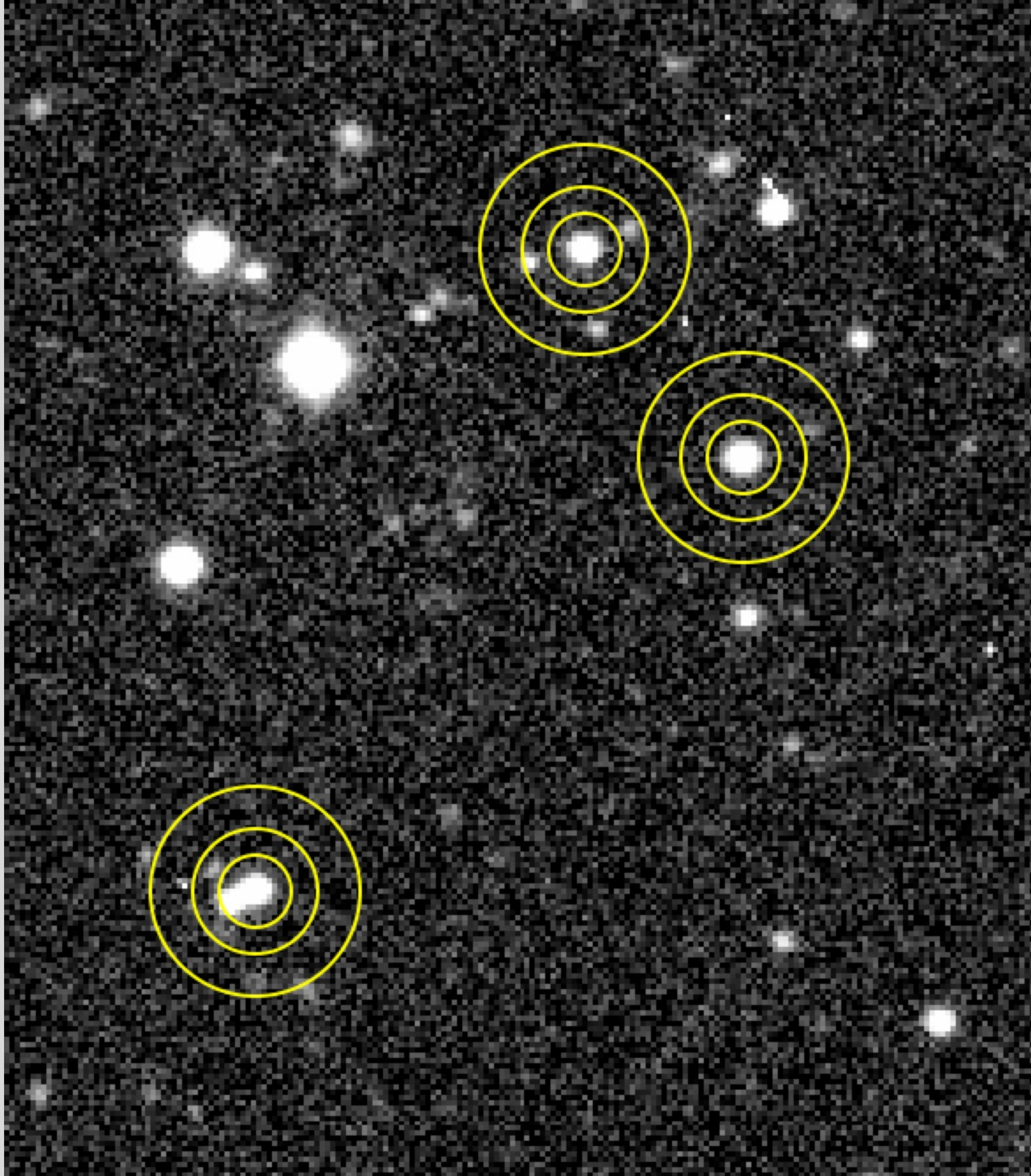
Aperture Photometry (Stars)

Measure flux (total counts) inside aperture of given size r, which contains n_{pix} pixels.

Estimate sky flux level (ADU/pix) from “average” ADU in pixels in surrounding annulus.

$$f = \left(\sum I_{aper}\right) - \langle I_{sky} \rangle \times n_{pix}$$

$$m = -2.5 \log f + (\text{calibration terms})$$



Aperture Photometry (Stars)

Apertures and aperture corrections

The bigger the aperture, the more flux and noise you get from sky photons in the aperture.

To maximize signal-to-noise, aperture should be dominated by star photons.

Optimal choice is $r_{ap} \approx$ full-width-at-half-max (FWHM) of the stellar profile.

This means the aperture is not collecting all the light from the star, and you need to correct for the missing light: **aperture corrections**

See HW #2 for the calculation....

aperture too big

aperture maximizing S/N

0.4" pixels, 1.2" FWHM seeing
Howell et al 1989

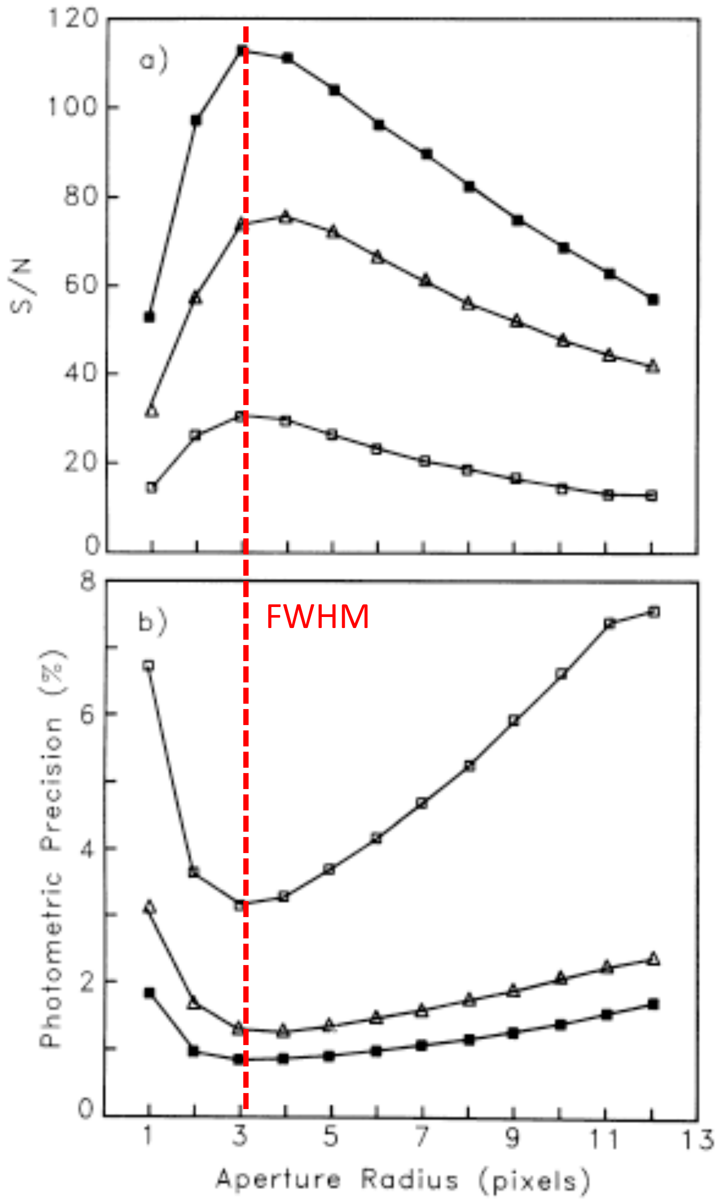
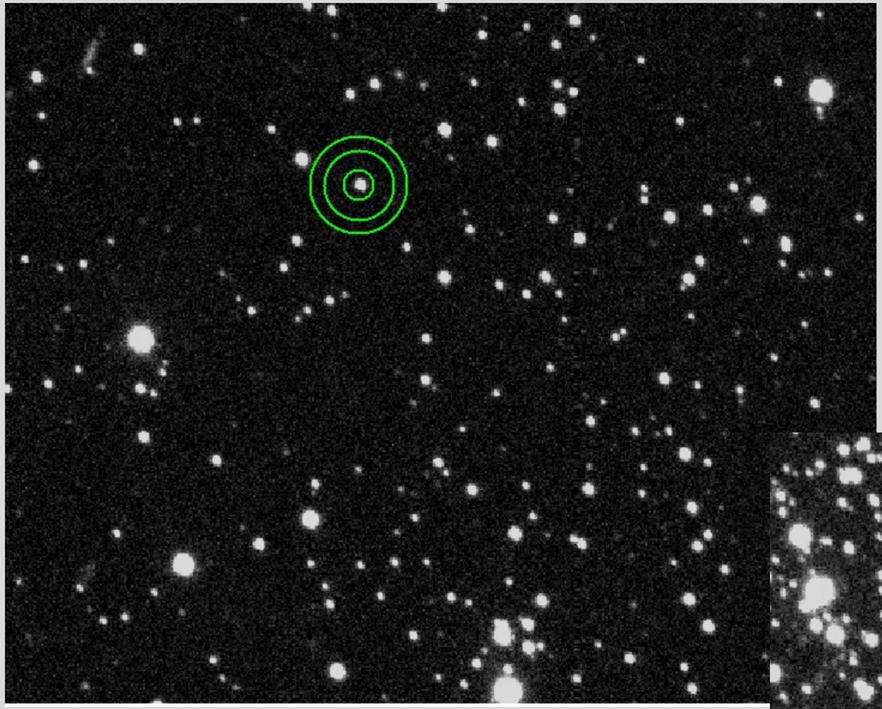
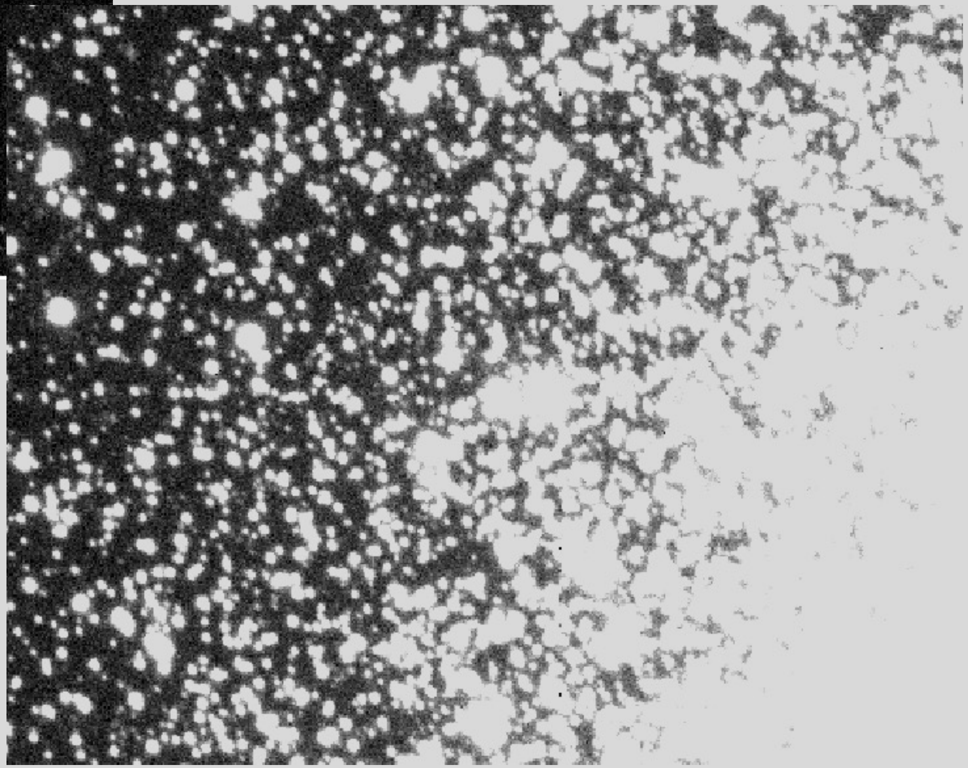


FIG. 6–S/N and photometric precision are plotted as functions of aperture radius for the same three point sources as before. The plots show that for a specific radius, which is fairly small, a point source has a maximum S/N and photometric precision. This maximum is not necessarily at the same radius for different objects. The symbols are ■; $V = 14.2$, △; $V = 14.5$, □; $V = 16.1$. The image scale is the same as in Figure 2.

PSF Fitting Photometry (Stars)



Aperture photometry good here.



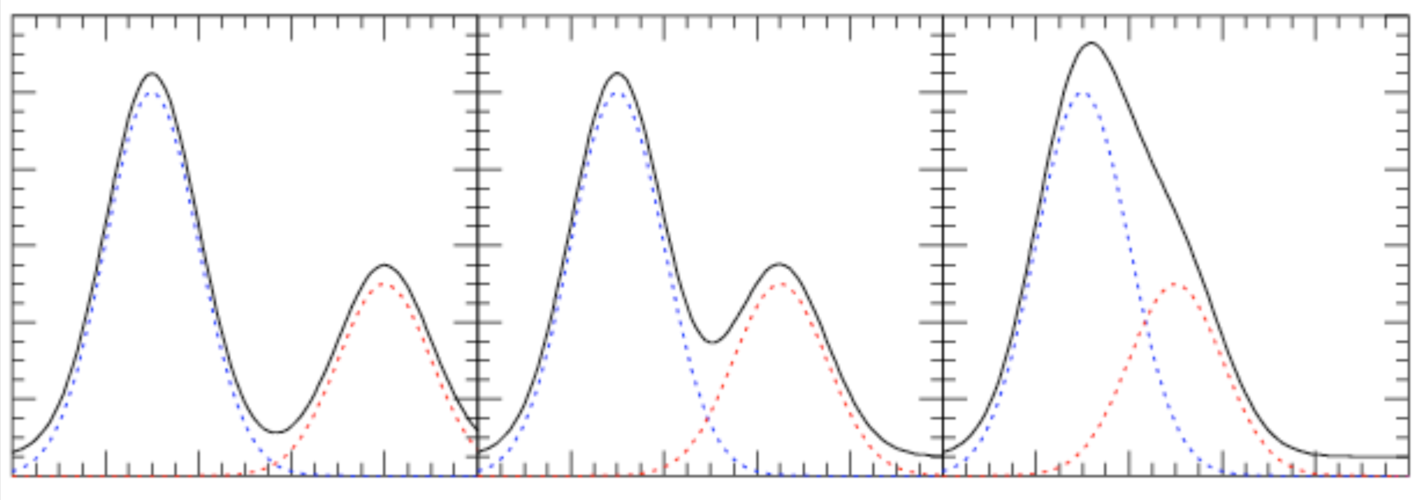
Aperture photometry not so good here!

PSF Fitting Photometry (Stars)

Model the 2D point spread function (PSF) based on bright (not saturated!) stars.

Fit the model to individual stars, varying brightness of model to minimize residuals.

1D analogy: fitting two Gaussians to a curve, easy to fit even with overlap



- Good for crowded fields.
- Often gives best magnitudes, as long as PSF model is well-determined
- But beware PSF variations (frame-to-frame, across the field of view, etc.)